Notes

2 septembre 2015

Chapitre 1

Logic and Proof

1.1 Objects

In mathematics you work with **Objects**.

Objects can be Real Numbers, Complex Numbers, Natural Numbers, etc.

With objects you create **Propositions**. We call a **logical proposition** with several mathematical objects a statement. The language of mathematics consists of primarily declarative sentences. If a sentence can be classified as true or false, it is called a **statement**.

Hence, statement can be either : **TRUE** or **FALSE**

TRUE is denoted as T or 1

FALSE is denoted as F or 0

Example 1.1.1. *9 is divisible by 4*

Answer : F, 0

When $n > 1$ then $n^2 \geqslant m$

Answer : T, 1

1.2 Connectors

Connectors are used to link logical propositions.

1.2.1   Types of Connectors

Name	Symbol	English
Disjunction	$\vee$	or
Conjuction	$\wedge$	and
implication	$\Rightarrow$	if...then
Negation	$\neg$	not
equivalance	$\Leftrightarrow$	iff

We can create *NEW* proposition with an old proposition by putting an operation/connector between them.

Example 1.2.1. *9 is divisible by 4 or 9 is divisible by 3.*

   Answer : T, 1

   9 is divisible by 4 and 9 is divisible by 3.

   Answer : F, 0

   9 is divisibly by 4 $\Rightarrow$ 9 is divisible by 3.

   Answer : T, 1

   9 is divisibly by 3 $\Rightarrow$ 9 is divisible by 4.

   Answer : F, 0

   9 is divisible by 3 $\Leftrightarrow$ 9 is divisibly by 4.

   Answer : F, 0

   (9 is divisible by 4.) $\neg$

   Answer : T, 1

   What does this statement equal ?

$$(A \text{ function is increasing}) \neg$$

Answer : *A function is NOT increasing.*

   *Note : ($\neg$ A function is increasing) $\nLeftrightarrow$ (A function is decreasing)
Because if a function is NOT increasing it does not mean that it is automatically decreasing. A function can be either increasing or decreasing.

1.2.2   Tables for Connectors

Disjunction

The Connective **or** is used to form a compound statement known as disjunction. In common English the word can have two meanings it can be either statement, but not both. In logic, there is a

possibility of it being both statements.

p	g	p ∨ g
T	T	T
T	F	T
F	T	T
F	F	F

Conjuction

The connective **and** is used in logic in the same way as it is in ordinary language. If p and q are statements, then the statement p and q are called the conjunction of p and q.

p	g	p ∧ g
T	T	T
T	F	F
F	T	F
F	F	F

Negation

Let p stand for a given statement. Then ¬ p (read as not p) represents the logical opposite (negation) of p.

p	¬ p
T	F
F	T

Implication

If **p, then q** is called an implication or a conditional statement. The if- statement p in the implication is called the **antecedent** and the then-statement q is called the **consequent**.

It is important to recognize that in mathematical writing the conditional statement can be disguised in several equivalent forms. Thus the following expressions all mean the same exact thing :

— if p, then q :   q provided that p
— p implies q :   q whenever p
— p only if q :   p is a sufficient condition for q
— q if p :   q is a necessary condition for p

p	g	p ⇒ g
T	T	T
T	F	F
F	T	T
F	F	T

Equivalance

A statement p **if and only if q** is the conjunction of two conditional statements p $\Rightarrow$ q and q $\Rightarrow p$. A statement of this form is a called a biconditional statement.

p	g	p $\Leftrightarrow$ g
T	T	T
T	F	F
F	T	F
F	F	T

1.3   Tautology

Tautology is when all the compound statements are true in all cases. When a biconditional statement is a tautology, it shows that the two parts of the biconditional are logically equivalent. That is, the two component statements have the same truth tables.

Statements that are logically equivalent :

$\neg$ (p $\wedge$ q) $\Leftrightarrow$ [($\neg$ p) $\vee$ ($\neg$ q)]

$\neg$ (p $\vee$ q) $\Leftrightarrow$ [($\neg$ p) $\wedge$ ($\neg$ q)]

$\neg$ (p $\Rightarrow$ q) $\Leftrightarrow$ [p $\wedge$ ($\neg$ q)]

[($\neg$ ($\neg$ p)]) $\Leftrightarrow$ p

($\neg$ p $\vee$ q) $\Leftrightarrow$ (p $\Rightarrow$ q)

($\neg$ p) $\Rightarrow$ ($\neg$ q) $\Leftrightarrow$ (p $\Rightarrow$ q)

(p $\Rightarrow$ q) $\wedge$ (q $\Rightarrow$ p) $\Leftrightarrow$ (p $\Leftrightarrow$ q)

1.3.1   Contrapositive

(p $\Rightarrow$ q) $\Leftrightarrow$ ($\neg$ q $\Rightarrow$ $\neg$ p)

1.3.2   Commutative Laws

(p $\wedge$ q) $\Leftrightarrow$ (q $\wedge$ p)

(p $\vee$ q) $\Leftrightarrow$ (q $\vee$ p)

1.3.3   Associative Laws

[p $\wedge$ (q $\wedge$ r)] $\Leftrightarrow$ [(p $\wedge$ q) $\wedge$ r]

[p $\vee$ (q $\vee$ r)] $\Leftrightarrow$ [(p $\vee$ q) $\vee$ r]

1.3.4   Distributive Laws

[p ∧ (q ∨ r)] ⇔ [(p ∧ q) ∨ (p∧ r)]

[p ∨ (q ∧ r)] ⇔ [(p ∨ q) ∧ (p ∨ r)]

1.4   Quantifiers

1.4.1   Propositions with quantifiers

Propositions can be expressed as a **function of variables.** Once a proposition involves a variable, it is customary to use functional notation. Hence, why we write p(x), which is a statement depending on x. You can also have more than one variable denoted as p(x,y). For a specific value of x, p(x) can be a statement that is either **True** or **False.** For this case, we will focus on one variable.

Example 1.4.1 (Example : x+y ⩾ 1). *Answer :This statement is True or False depending on which value you assign to x and y.*

Lets denote E as the set we are working with.

1. "∀" x ∈ E, p(x) ⇔ "For any.... For every... For all... For each... x ∈ E, p(x) is true."

2. "∃" x ∈ E, p(x) ⇔ "There is a... There exist... There is at least one... x ∈ E such that p(x) is true."

The words "no" and "none" are universal quantifiers in a negative sense. In general, the statement "None of them are p(x)" is equivalent to "All of them are not p(x)"

Remarks

(1) Is true *iff* each value in x in E makes p(x) true.

(2) is true *iff* if there is one value x which makes it true.

"∀" x ∈ E, p(x) ⇔ "∀" y∈ E, p(y)

The only thing that is different here is the variables. Hence, the propositions mean the same thing.

It is very important to say where x lies because considering an equation in complex numbers or real numbers changes completely the value of the proposition.

If x is greater than 1, then $x^2$is greater than 1. The intended meaning is : ∀ x∈ ℝ, if x > 1, then $x^2 > 1$. This gives you a different answer if x ∈ ℂ.

It is important to realize that the order in which quantifiers are used affects the truth value.
For example, when talking about real numbers, the statement
  — ∀ x ∈ ℝ, ∃ y ∈ ℝ such that y >x.
   Is true. That is, given any real number x there is always a real number y that is greater than x.
   But the statement

— $\exists$ y $\in$ $\mathbb{R}$ such that $\forall$ x $\in$ $\mathbb{R}$, y > x.
Is false, since there is no fixed real number y that is greater than every real number. Thus care must be taken when reading and writing quantified statements so that the order of the quantifiers is not inadvertently changed.

1.4.2   Negation

1. "$\forall$" x $\in$ E, p(x) $\Leftrightarrow$ "$\exists$" x $\in$ E such that $\neg$ p(x)

2. "$\exists$" x $\in$ E such that p(x) $\Leftrightarrow$ "$\forall$" x $\in$ E, $\neg$ p(x)

Note : We do not change the set E nor the symbol $\in$.

Proving Quantifiers :

— To prove that "$\exists$" x $\in$ E such that p(x) is true it is enough to find a precise example of value x, which makes p(x) true

To prove that "$\forall$" x $\in$ E, p(x)" is true the technique above does NOT work. Even if you give plenty of values which make it true it could be the remaining ones that fail.

— To prove $\forall$"x $\in$ E, p(x)" :
Let x $\in$ E,
Continues with a more logical proof
P(x) is true

1.5   Proof techniques

1.5.1   Inductive Reasoning vs. Deductive Reasoning

Inductive Reasoning- Drawing a conclusion based on examples. Although it is true that some of the conclusions drawn from inductive reasoning are not true, they can be very useful.

Deductive Reasoning- Applying a general principle to a particular case. Mathematics is based on this type of reasoning.

If you want to prove that the statement "$\forall$" x $\in$ E, p(x) " is false you need to prove, "$\exists$ x $\in$ E, which makes p(x) false." It is enough to do this. This is called a **Counterexample.** A counterexample is an example that makes a statement false.

The most common type of mathematical thereom can be symbolized as p $\Rightarrow$ q. p $\Rightarrow$ q is a tautology. As a result, in order to prove that p $\Rightarrow$ q we must show that whenever p is true it follows that q must be true. p is called the **Hypothesis** and q is called the **Conclusion.**

The construction of a proof of the implication p $\Rightarrow$ q can be called a building bridge consisting of logical statements. The goal is to connect the hypothesis p with the conclusion q.

Ways that p and q can be connected :

1. Definitions

2. Assumptions or axioms that are accepted as true

3. Theorems that have been previously established as true

4. Statements that are logically implied by the earlier statements in the proof

1.5.2 Contrapositive

If the implication is p $\Rightarrow$ q the contrapositive is $\neg$ q $\Rightarrow$ $\neg$ p

Let p and q be two propositions :

— Suppose we want to prove that p $\Rightarrow$ q is true.

By the principle of contrapositive we know that it is equivalent to prove that $(\neg$ q$) \Rightarrow (\neg$ p$)$ is true.
In essence, (p $\Rightarrow$ q) $\Leftrightarrow$ ($\neg$ q $\Rightarrow$ $\neg$ p)

Example 1.5.1. *Let x=p/q a rational number not zero (pq $\neq$ 0) We will prove by contra-*
positive that $\forall$ y $\in \mathbb{R} \setminus \mathbb{Q} \Rightarrow xy \in \mathbb{R} \setminus \mathbb{Q}$)
The contrapositive of this is : $\forall$ y $\in \mathbb{R}$, (xy $\in \mathbb{Q} \Rightarrow$ y $\in \mathbb{Q}$)
Proof :
Let y $\in \mathbb{R}$
xy= a/b
We do not have b $\neq$ 0
y= a/b $\times$ 1/x
y= a/b $\times$ q/p
We are supposing x=p/q
y= aq/bp $\in \mathbb{Q}$
By the contrapositive, we have proven that $\forall$ y $\in \mathbb{R}$ (y $\in \mathbb{R} \setminus \mathbb{Q} \Rightarrow$ xy $\in \mathbb{R} \setminus \mathbb{Q}$)

1.5.3 Converse

If the implication is p $\Rightarrow$ q the converse is q $\Rightarrow$ p

1.5.4 Inverse

If the implication is p $\Rightarrow$ q the inverse is $\neg$ p $\Rightarrow$ $\neg$ q

1.5.5 Contradiction

Mathematical theorems and proofs do not occur in isolation, but always in the context of some mathematical system. Therefore, it is really important to know where your variable lives. For example, does your variable live in the real numbers, complex numbers, integers, etc.

In some situations it is possible to prove an existential statement in an indirect way without actually producing any specific member of the system. One indirect method is

to use the contrapositive side of the implication and another is to used a proof by contradiction.

A reasoning by contradiction is a technique of proof which is based on the fact that
$p \vee \neg p$ is a tautology.

Here is how to do it :

Principle :
Suppose we want to prove that a proposition p is true
We suppose that $\neg$ (p) is true (p false) and we exhibit (via a system of axioms, rules, logic deduction...)
a contradiction so we can conclude that the hypothesis on p was wrong and then that p is true.

The two basic forms of a proof by contradiction are based on tautologies.

$(\neg p \Rightarrow c) \Leftrightarrow p$
If we wish to conclude a statement p we can do so by showing that the negation of p leads to a contradiction c. (Which is what was explained above)

$(p \Rightarrow q) \Leftrightarrow [(p \wedge \neg q) \Rightarrow c$
If we wish to conclude that p implies q, we can do so by showing that p and not q leads to a contradiction.

In order to proof $(p \vee q) \Rightarrow r$
We can prove $(p \Rightarrow r) \wedge (q \Rightarrow r)$